Linear Ordinary Differential Equations: breaking through the order 2 barrier

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Abstract

A major subproblem for algorithms that either factor ordinary linear differential equations or compute their closed form solutions is to find their solutions y which satisfy $y'/y \in \overline{K}(x)$ where K is the constant field for the coefficients of the equation. While a decision procedure for this subproblem was known in the 19^{th} century, it requires factoring polynomials over $\overline{K}$ and has not been implemented in full generality. We present here an efficient algorithm for this subproblem, which has been implemented in the AXIOM computer algebra system for equations of arbitrary order over arbitrary fields of characteristic 0. This algorithm never needs to compute with the individual complex singularities of the equation, and algebraic numbers are added only when they appear in the potential solutions. Implementation of the complete Singer algorithm for $n = 2, 3$ based on this building block is in progress.

1 Introduction

Consider the linear ordinary differential equation

$$A_n(x)y^{(n)} + \ldots + A_1(x)y' + A_0(x)y = 0 \qquad (1)$$

where the A_i's are polynomials with coefficients in some constant field K. The problem of finding whether (1) has a "closed-form" solution has been of great interest to mathematicians since the second half of the 19^{th} century. While the problem of finding all the Liouvillian solutions of (1) is decidable in theory for any order ([8]), the published decision procedure is not considered a practical algorithm and has not been implemented. Currently, even the most advanced computer algebra systems are limited to the second order case ($n = 2$), for which they use Kovacic's algorithm ([6]). Even for that case, implementations are usually incomplete and authors have reported difficulties in dealing with arbitrary complex singularities ([11]). These difficulties usually stem from the need of those algorithms to compute with the singularities of (1) in $\overline{K}$.

We describe here an efficient algorithm, fully implemented in the AXIOM computer algebra system, that returns a basis over $\overline{K}$ for the space of the solutions y of (1) such that $y'/y \in \overline{K}(x)$ (we call them the *exponential solutions of (1) over* $\overline{K}$ in the rest of this paper). The algorithm is "rational" in the sense that algebraic elements over K are introduced only if they appear in the potential solutions, and not in the singularities of the equation as was previously required. Thus, we do not need to factor $A_n(x)$, even over K.

While the exponential solutions form only a subspace of the Liouvillian solutions, the main subalgorithm of the algorithms that find the Liouvillian solutions ([6, 8]) is to find the exponential solutions of higher order equations. So, as applications of the algorithm described here we get:

- a "rational" version of Kovacic's algorithm for $n = 2$,
- with the bounds of [10], an efficient version of Singer's algorithm for $n = 3$,
- a version of the linear differential operator factoring algorithm of [7] which factors over $\overline{K}(x)$ (the existing implementation of that algorithm factors over $\mathbf{Q}(x)$ only).

All of those applications are currently being implemented [1]. In addition, the search for the exponential solutions alone is being used as an incomplete n^{th}-order

[1] The last one in conjunction with F.Schwarz at the GMD, Bonn

solver in the AXIOM computer algebra system, and is successful on a large number of higher order equations from [5] and other sources.

2 Rational solutions

A solution y of (1) is called *rational* if $y \in \overline{K}(x)$. The rational solutions form a finite dimensional vector space $\overline{V}$ over $\overline{K}$. Let V be the vector space generated over K by the solutions of (1) which are in $K(x)$. By Theorem 9.1 of [4], $dim_K(V) = dim_{\overline{K}}(\overline{V})$ and a basis for V over K is also a basis for $\overline{V}$ over $\overline{K}$, so it is sufficient to look for the rational solutions in $K(x)$. A fast rational algorithm that finds a basis for V over K is described in [1, 2, 3] and has been implemented in the computer algebra systems AXIOM, Maple and REDUCE, so we do not repeat its description here. Its main innovation is the use of *balanced factorisation* which we outline here since it will be used in the next part of the algorithm.

Recall that for $P, Q \in K[x] \setminus \{0\}$, $\deg(P) > 0$, the *order* of Q at P, denoted $\nu_P(Q)$ is the unique integer m such that $P^m \mid Q$ and $P^{m+1} \nmid Q$.

Definition 1 *Let $A, B \in K[x]$. We say that A is* balanced w.r.t. B *if $\nu_P(B) = \nu_Q(B)$ for any two irreducible factors P and Q of A in $K[x]$. Let $\mathcal{S} \subseteq K[x]$. We say that $A = A_1^{e_1} \cdots A_m^{e_m}$ is a* balanced factorisation of A *w.r.t $\mathcal{S}$ if each A_i is squarefree and balanced w.r.t B for any $B \in \mathcal{S}$, and $(A_i, A_j) = (1)$ for $i \neq j$.*

In other words, a polynomial A is balanced w.r.t. to B if all the irreducible factors of A appear with the same power in B. A balanced factorisation of A w.r.t. a set of polynomials $\mathcal{S}$ is a factorisation of A into squarefree coprime factors, each of which is balanced w.r.t. each element of $\mathcal{S}$.

An algorithm for computing balanced factorisations appeared in [1] and slightly modified in [3]. The interesting point computationally is that it only requires gcd computations in $K[x]$, and is at most as expensive as computing a square-free-gcd-free basis for $\{A\} \cup \mathcal{S}$.

3 The associated Riccati equation

In order to look for the other exponential solutions of (1) over $\overline{K}$, we consider the *associated Riccati equation* defined as follow: let u be a new differential indeterminate and define the sequence of differential polynomials $(P_i)_{i \geq 0}$ by

$$\left\{ \begin{array}{lcl} P_0 & = & 1 \\ P_i & = & {P_{i-1}}' + uP_{i-1} \quad \text{for } i > 0. \end{array} \right. \tag{2}$$

i.e. $P_0 = 1, P_1 = u, P_2 = u' + u^2, P_3 = u'' + 3uu' + u^3$, etc...

Definition 2 *The* associated Riccati equation *of (1) is the non-linear ordinary differential equation:*

$$A_n(x)P_n + \ldots + A_1(x)P_1 + A_0(x)P_0 = 0. \tag{3}$$

Example: The associated Ricatti equation of $y'' = ry, r \in K(x)$ (the equation solved by Kovacic's algorithm) is $u' + u^2 = r$, which is a classical Ricatti equation.

The key property of the associated Riccati equation, which is proven by substituting $e^{\int u}$ for y in equation (1) is: *if $y \neq 0$ is any solution of (1), then $u = y'/y$ is a solution of (3); conversely, if u is any solution of (3), then $y = e^{\int u}$ is a solution of (1).*

Since for an exponential solution y of (1), $u = y'/y \in \overline{K}(x)$, finding the exponential solutions of (1) is equivalent to finding the rational solutions of (3).

4 Rational solutions of the Riccati equation

We now look for the solutions u of (3) which are in $\overline{K}(x)$. In opposition to the linear case, it can happen that there are solutions in $\overline{K}(x)$ and none in $K(x)$. For example the associated Riccati equation of $y'' + y = 0$ is $u' + u^2 = -1$ which has the 2 solutions $\pm\sqrt{-1}$ but no solution in $\mathbf{Q}(x)$. So it may be necessary to extend K at some point in the algorithm.

In order to bound the order of the singularities of a solution of (3), we need the following construction which is somewhat similar to the Newton polygon of equation (1) at a singularity:

Definition 3 *Let $\mathcal{S} = \{Q_1, \ldots, Q_m\} \subset K[x]$. We define the* Riccati exponents *of $\mathcal{S}$ at ∞ to be:*

$$\Gamma_\infty(\mathcal{S}) = \{\frac{\deg(Q_i) - \deg(Q_j)}{i - j} \text{ for } 1 \leq i \neq j \leq n$$

$$\text{such that } Q_i \neq 0 \neq Q_j\} \cap \mathbf{N} \setminus \{0\}$$

where $\mathbf{N}$ is the set of the nonnegative integers, and the Riccati bound *of $\mathcal{S}$ at ∞ to be:*

$$\delta_\infty(\mathcal{S}) = \left\{ \begin{array}{cl} 0 & \text{if } \Gamma_\infty(\mathcal{S}) = \emptyset \\ \max(\Gamma_\infty(\mathcal{S})) & \text{otherwise.} \end{array} \right.$$

Similarly, for $P \in K[x] \setminus \{0\}$, $\deg(P) > 0$, we define the Riccati exponents *of $\mathcal{S}$ at P to be:*

$$\Gamma_P(\mathcal{S}) = \{\frac{\nu_P(Q_i)-\nu_P(Q_j)}{i-j} \text{ for } 1 \le i \ne j \le n \text{ such that } Q_i \ne 0 \ne Q_j\} \cap \mathbf{N} \setminus \{0\}$$

and the Riccati bound *of $\mathcal{S}$ at P to be:*

$$\delta_P(\mathcal{S}) = \begin{cases} 1 & \text{if } \Gamma_P(\mathcal{S}) = \emptyset \\ \max(\Gamma_P(\mathcal{S})) & \text{otherwise.} \end{cases} \tag{4}$$

Our algorithm is based on the algorithm of [9], but uses the following Theorem to avoid factoring A_n. This Theorem is essentially the base case ($' = d/dx$) of Theorem 8.4 of [4]:

Theorem 1 *Let $A_0,\dots,A_n \in K[x]$ with $A_n \ne 0$ and monic. Let $C_1^{e_1}\dots C_q^{e_q}$ be a balanced factorisation of A_n w.r.t. $\mathcal{S} = \{A_0,\dots,A_n\}$. Then, any solution $u \in \overline{K}(x)$ of equation (3) can be written as*

$$u = \frac{B_1}{C_1^{\delta_{C_1}(\mathcal{S})}} + \dots + \frac{B_q}{C_q^{\delta_{C_q}(\mathcal{S})}} + \frac{Q'}{Q} + P \tag{5}$$

where $\delta_{C_i}(\mathcal{S})$ is given by (4), $Q, P, B_1, \dots, B_q \in \overline{K}[x]$, $(Q, A_n) = (1)$ and $\deg(B_i) < \delta_{C_i}(\mathcal{S})\deg(C_i)$ for each i.

The algorithm proceeds by computing a finite set of possibilities for each summand of (5). Everytime an additive component α is found, we do the change of variable $u = \alpha + \overline{u}$ in equation (3) and obtain a new Ricatti equation for $\overline{u}$. We show first how such a change of variable can be done by computing with operators over $K(x,\alpha)$.

4.1 Additive change of variable in the Ricatti equation

Suppose that we have found a component $\alpha \in \overline{K}(x)$ of the sum (5). We set $u = \alpha + \overline{u}$ and want a new Ricatti equation for $\overline{u}$. Let $E = K(x,\alpha)[\partial_x]$ be the ring of the linear ordinary differential operators over $K(x,\alpha)$ where ∂_x is differentiation with respect to x (such a ring is a provided as a type in AXIOM, and is easy to implement in the other systems). The operator corresponding to equation (1) is $L = \sum_{i=0}^n A_i \partial_x^i$. Let $y = e^{\int u}$ and $\overline{y} = e^{\int \overline{u}}$. Since $u = \alpha + \overline{u}$, we have $y = e^{\int \alpha}\overline{y}$. Also, if u is a solution of (3), then $Ly = 0$, so $L(\overline{y}e^{\int \alpha}) = 0$. But, from the arithmetic in E, $L(\overline{y}e^{\int \alpha}) = e^{\int \alpha}\overline{L}\overline{y}$ where $\overline{L} = \sum_{i=0}^n A_i(\partial_x + \alpha)^i$. So the Ricatti equation satisfied by $\overline{u}$ is just the associated Ricatti equation of $\overline{L}$, and the change of variable procedure is:

change_var$(A_0,\dots,A_n,\alpha)$
OUTPUT: $C_0,\dots,C_n$ such that when $u = \alpha + \overline{u}$ is a solution of (3), $\overline{u}$ is a solution of $\sum_{i=0}^n C_i P_i = 0$.

- $\overline{L} \leftarrow \sum_{i=0}^n A_i(\partial_x + \alpha)^i \in K(x,\alpha)[\partial_x]$
- return [coefficient of ∂_x^i in $\overline{L}$ for $i = 0,\dots,n$]

For the case where arithmetic in E is not readily available, there is a closed-form formula for the operator $\overline{L}$ of the first step, namely

$$(\partial_x + \alpha)^i = \sum_{j=0}^i \binom{i}{j} P_{i-j}(\alpha)\partial_x^j$$

where the $P_k(\alpha)$'s are given by the recursion (2) with u replaced by α.

4.2 The polynomial part

We introduce the notation $lc(P)$ for the leading coefficient of a polynomial P, and we use the notation of Theorem 1 in the rest of this paper. From the proof of Proposition 2.3 of [9] we have $\deg(P) \le \delta_\infty(\mathcal{S})$, and if $\deg(P) = m$, then $H_m(lc(P)) = 0$ where $H_m \in K[Y]$ is given by:

$$d_m \leftarrow \max_{\substack{0 \le i \le n \\ A_i \ne 0}} (im + \deg(A_i))$$

$$H_m = \sum_{\substack{0 \le i \le n \\ A_i \ne 0 \\ im + \deg(A_i) = d_m}} lc(A_i)Y^i. \tag{6}$$

So the potential leading coefficients of P are the roots of H_m, and the algorithm that returns all the potential polynomial parts in (5) is:

poly_part$(A_0,\dots,A_n)$
OUTPUT: A set $\{P_1,\dots,P_t\}$ of polynomials such that any solution $u \in \overline{K}(x)$ of (3) can be written in the form (5) with $P = P_i$ for some i.

- return
  poly_part_bounded$(A_0,\dots,A_n, \delta_\infty(\{A_0,\dots,A_n\}))$

poly_part_bounded$(A_0,\dots,A_n,m)$

- $ans \leftarrow \{0\}$
- **if** $m < 0$ **then** return ans
- $H_m \leftarrow$ polynomial given by (6)
- **for** $a \in \overline{K} \mid H_m(a) = 0$ **do**
  $\alpha \leftarrow ax^m$
  $\{D_0,\dots,D_n\} \leftarrow$ **change_var**$(A_0,\dots,A_n,\alpha)$
  $\{P_1,\dots,P_t\} \leftarrow$
  poly_part_bounded$(D_0,\dots,D_n,m-1)$
  $ans \leftarrow ans \cup \{\alpha + P_1,\dots,\alpha + P_t\}$
- return ans

4.3 The P-adic part

Notation: for $A, B \in K[x]$, $B \neq 0$, we write $\phi_B(A)$ for the remainder of A modulo B.
Let C be one of the C_i's of Theorem 1, δ be $\delta_C(\mathcal{S})$ and rewrite the term R/C^δ of the sum (5) as

$$\frac{R}{C^\delta} = \frac{R_\delta}{C^\delta} + \ldots + \frac{R_1}{C} \tag{7}$$

where $\deg(R_i) < \deg(C)$. We call (7) the *P-adic part of u at C*. From the proofs of Theorem 8.4 of [4] and Proposition 2.3 of [9], we have

$$H_\delta(R_\delta) \equiv 0 \pmod{C} \tag{8}$$

where $H_\delta \in K[x][Y]$ is given by

$$n_\delta \leftarrow \max_{\substack{0 \leq i \leq n \\ A_i \neq 0}} (i\delta - \nu_C(A_i))$$

$$H_\delta = \sum_{\substack{0 \leq i \leq n \\ A_i \neq 0 \\ i\delta - \nu_C(A_i) = n_\delta}} \phi_C\left(\frac{A_i}{C^{\nu_C(A_i)}}\right) S_{\delta,i}(Y). \tag{9}$$

where

$$S_{\delta,i}(Y) = \begin{cases} Y^i & \text{if } \delta > 1 \\ \prod_{j=0}^{i-1}(Y - jC') & \text{if } \delta = 1. \end{cases}$$

So the potential leading coefficients of (7) are the solutions of (8) in $\overline{K}[x]$ of degree at most $\deg(C)-1$. There are two alternatives for solving equation (8):

1. Since $deg(R_\delta) < \deg(C)$ we write $R_\delta = a_0 + a_1 x + \ldots a_{m-1}x^{m-1}$ where $m = \deg(C)$ and the a_i's are indeterminates. Plugging into equation (8), reducing modulo C and equating to 0, we get a zero-dimensional non-linear algebraic system for $a_0, \ldots, a_{m-1}$, and we use the existing non-linear algebraic solvers to find all the solutions in $\overline{K}$.
2. The algorithm of [9] factors C into irreducibles over K, then for each factor Q_i, it factors $H_\delta(\gamma)(Y)$ over $\overline{K}$ where $Q_i(\gamma) = 0$. For each linear factor $Y - S(\gamma)$ of $H_\delta(\gamma)(Y)$, $R = S(x)$ is a potential solution of (8).

Our implementation uses the first method (non-linear algebraic system), since Gröbner basis implementations are available over arbitrary fields and solvers are now reasonably efficient for zero-dimensional systems with a small number of solutions (which are the systems arising when the initial differential equation has a small order). Depending on the actual constant field in each case, it can payoff to factor C further thereby reducing the number of variables, but testing with examples from [5] has not shown the need for factoring C.

With a solver for equation (8) available, the algorithm for computing all the potential P-adic parts at C in (5) parallels exactly the algorithm for the polynomial part (it is in fact the same if one considers the degree function to be the negative of the order at infinity):

padic_part$(A_0, \ldots, A_n, C)$
OUTPUT: A set $\{f_1, \ldots, f_t\}$ of fractions such that any solution $u \in \overline{K}(x)$ of (3) can be written in the form (5) with $B/C^{\delta_C(\mathcal{S})} = f_i$ for some i.

- return
  padic_part_bounded$(A_0, \ldots, A_n, C,$
  $\delta_C(\{A_0, \ldots, A_n\}))$

padic_part_bounded$(A_0, \ldots, A_n, C, \delta)$

- $ans \leftarrow \{0\}$
- **if** $\delta < 1$ **then** return ans
- $H_\delta \leftarrow$ polynomial given by (9)
- **for** $R \in \overline{K}[x] \mid \deg(R) < \deg(C) \& H_\delta(R) \equiv 0(C)$ **do**
  $\alpha \leftarrow R/C^m$
  $\{D_0, \ldots, D_n\} \leftarrow$ **change_var**$(A_0, \ldots, A_n, \alpha)$
  $\{f_1, \ldots, f_t\} \leftarrow$
  padic_part_bounded$(D_0, \ldots, D_n, C, \delta - 1)$
  $ans \leftarrow ans \cup \{\alpha + f_1, \ldots, \alpha + f_t\}$
- return ans

Once the individual candidates for each P-adic part can computed, finding the entire singular part of (5) (except for the Q'/Q term) is done by computing all the potential P-adic parts at each balanced factor C_i and grouping them:

singular_part$(A_0, \ldots, A_n)$
OUTPUT: A set $\{f_1, \ldots, f_t\}$ of fractions such that any solution $u \in \overline{K}(x)$ of (3) can be written in the form $f_i + Q'/Q + P$ with Q and P as in (5).

- $C_1^{e_1} \ldots C_q^{e_q} \leftarrow$
  balanced factorisation of A_n w.r.t. $\{A_0, \ldots, A_n\}$
- return **singular_part1**$(A_0, \ldots, A_n, C_1, \ldots, C_q, q)$

singular_part1$(A_0, \ldots, A_n, C_1, \ldots, C_m, m)$

- $ans \leftarrow \{0\}$
- **if** $m < 1$ **then** return ans

- $\{f_1,\ldots,f_t\} \leftarrow$ **padic_part**$(A_0,\ldots,A_n,C_m)$
- **for** $i = 1\ldots t$ **do**
  $\{D_0,\ldots,D_n\} \leftarrow$ **change_var**$(A_0,\ldots,A_n,f_i)$
  $\{g_1,\ldots,g_s\} \leftarrow$
  singular_part1$(D_0,\ldots,D_n,$
  $C_1,\ldots,C_{m-1},m-1)$
  $ans \leftarrow ans \cup \{f_i+g_1,\ldots,f_i+g_s\}$
- return ans

4.4 The logarithmic derivative part

We can now compute a finite set of possibilities for all the quantities appearing in (5) except for the Q'/Q term. For each possibility α we do the change of variable $u = \alpha + Q'/Q$ and obtain a Ricatti equation $R = \sum_{i=0}^n D_i P_i$ such that $R(Q'/Q) = 0$. But, since $e^{\int Q'/Q} = Q$, we get $L(Q) = 0$ where $L = \sum_{i=0}^n D_i \partial_x^i$. We can find a basis $\{f_1,\ldots,f_z\}$ for the rational solutions of $Ly = 0$ by the algorithm of section 2. If this basis is empty, then equation (3) has no solution with this particular α, otherwise $\{\alpha+f_1,\ldots,\alpha+f_z\}$ are all solutions of (3). After grouping all such solutions for the various possible α's we get all the solutions of (3), hence a basis for the exponential solutions of (1).

solve_ricatti$(A_0,\ldots,A_n)$
OUTPUT: The set of all the solutions of equation (3) in $\overline{K}(x)$.

- $ans \leftarrow \{0\}$
- $\{P_1,\ldots,P_t\} \leftarrow$ **poly_part**$(A_0,\ldots,A_n)$
- **for** $i = 1\ldots t$ **do**
  $\{D_0,\ldots,D_n\} \leftarrow$ **change_var**$(A_0,\ldots,A_n,P_i)$
  $\{g_1,\ldots,g_s\} \leftarrow$ **singular_part**$(D_0,\ldots,D_n)$
  for $j = 1\ldots s$ **do**
  $\{F_0,\ldots,F_n\} \leftarrow$ **change_var**$(D_0,\ldots,D_n,g_j)$
  $\{h_0,\ldots,h_z\} \leftarrow$
  basis for the rational solutions
  of $\sum_{i=0}^n F_i y^{(i)} = 0$
  $ans \leftarrow ans \cup \{f_i+g_j+{h_1}'/h_1,\ldots,$
  $f_i+g_j+{h_z}'/h_z\}$
- return ans

5 Example

Consider the equation

$$(x^2-x^4)y'' - (8x^3-5x+1)y' + (3-12x^2)y = 0 \quad (10)$$

which has no rational solution. Its associated Ricatti equation is

$$(x^2-x^4)(u'+u^2) - (8x^3-5x+1)u + 3 - 12x^2 = 0. \quad (11)$$

A balanced factorisation of $A_2 = x^2 - x^4$ w.r.t. $\mathcal{S} = \{A_0, A_1, A_2\}$ is $A_2 = C_1 C_2^2 = (1-x^2)x^2$, and the order information of the coefficients of (10) is in the following table:

i	A_i	$\nu_x(A_i)$	$\nu_{1-x^2}(A_i)$	$\deg(A_i)$
0	$3-12x^2$	0	0	2
1	$-8x^3+5x-1$	0	0	3
2	x^2-x^4	2	1	4

(12)

The polynomial part: From (12), $\Gamma_\infty(\mathcal{S}) = \{1,2\}$ so $\delta_\infty(\mathcal{S}) = 2$. By formula (6), $d_2 = \max_{i=0,1,2}(2i + \deg(A_i)) = 8$, and $H_2 = lc(A_2)Y^2 = -Y^2$. The only root of H_2 is 0, so the degree of the polynomial part of a rational solution of (11) is at most 1. Again, by formula (6), $d_1 = \max_{i=0,1,2}(i + \deg(A_i)) = 6$, and $H_1 = lc(A_2)Y^2 = -Y^2$, so the polynomial part must have degree 0. By formula (6) once more, $d_0 = \max_{i=0,1,2}(\deg(A_i)) = 4$, and $H_0 = lc(A_2)Y^2 = -Y^2$, so any rational solution of (11) must be of the form (5) with $P = 0$.
The P-adic part: From (12), $\Gamma_x(\mathcal{S}) = \{1,2\}$, so $\delta_x(\mathcal{S}) = 2$. By formula (9), $n_2 = \max_{i=0,1,2}(2i - \nu_x(A_i)) = 2$, and

$$H_2 = \phi_x(\frac{A_2}{x^2})Y^2 + \phi_x(A_1)Y = Y^2 - Y$$

so we look for the solutions R of degree 0 in $\overline{K}[x]$ of $R^2 - R \equiv 0 \pmod{x}$. Since $\deg(R) = 0$, we set $R = a \in \overline{K}$, and solve $a^2 - a = 0$ in $\overline{K}$. The 2 solutions are $a = 0$ and $a = 1$, so the 2 possible terms of order -2 at $x = 0$ are 0 and $1/x^2$. We must look for the term of order -1 in both of those cases.
Case $a = 0$: Setting $\delta = 1$, we have by formula (9): $n_1 = \max_{i=0,1,2}(i - \nu_x(A_i)) = 1$, and $H_1 = \phi_x(A_1)Y = -Y$, which has only 0 for solution, so the only possible P-adic part at x in this case is 0.
Case $a = 1$: We must first do the change of variable $u = 1/x^2 + \overline{u}$, and the Ricatti equation for $\overline{u}$ is the one associated with

$$\overline{L} = A_2(\partial_x + \frac{1}{x^2})^2 + A_1(\partial_x + \frac{1}{x^2}) + A_0$$
$$= -\frac{B_2\partial_x^2 + B_1\partial_x + B_0}{x}$$

where $B_2 = x^5 - x^3$, $B_1 = 8x^4 + 2x^3 - 5x^2 - x$ and $B_0 = 12x^3 + 6x^2 - 2x - 3$. We first look for the term of order -1 at x of $\overline{u}$. From formula (9), $n_1 = \max_{i=0,1,2}(i - \nu_x(B_i)) = 0$, and $H_1 = \phi_x(B_1/x)Y = -Y$, so the the only possible P-adic part for $\overline{u}$ at x is 0.
The last remaining P-adic part for $\overline{u}$ is at $C_1 = 1 - x^2$, and we find that $\Gamma_{1-x^2}(\{B_0,B_1,B_2\}) = \{1\}$, so $\delta_{1-x^2}(\{B_0,B_1,B_2\}) = 1$. By formula (9), $n_1 = \max_{i=0,1,2}(i - \nu_{1-x^2}(B_i)) = 1$, and

$$H_1 = \phi_{1-x^2}(\frac{B_2}{1-x^2})Y(Y+2x) + \phi_{1-x^2}(B_1)Y$$
$$= -xY(Y+2x) + (x+3)Y$$
$$= -xY^2 - (2x^2 - x - 3)Y$$

so we look for the solutions R of degree 1 in $\overline{K}[x]$ of $-xR^2 - (2x^2 - x - 3)R \equiv 0 \pmod{1-x^2}$, which reduces to $-xR^2 + (x+1)R \equiv 0 \pmod{1-x^2}$. Since $\deg(R) = 1$, we set $R = a + bx, a, b \in \overline{K}$, plug in the above equation, reduce modulo $1 - x^2$ and we get the following system for a and b:

$$a + b - 2ab = -a^2 - b^2 + a + b = 0$$

which has the 2 solutions $a = b = 0$ and $a = b = 1$. So the P-adic expansion of $\overline{u}$ at $1 - x^2$ is either 0 or $(1+x)/(1-x^2) = 1/(1-x)$. So the singular part of $\overline{u}$ is either 0 or $1/(1-x)$, so the singular part of u is either 0, $1/x^2$ or $1/x^2 + 1/(1-x)$. Since u has no polynomial part, we are left with computing the Q'/Q part for each of those 3 cases.

The logarithmic derivative part: We have 3 cases, depending on the potential singular part of u.

Case 0*:* Then, since (10) has no nonzero rational solution, (11) has no rational solution.

Case $1/x^2$*:* Then, we check whether $B_2y'' + B_1y' + B_0 = 0$, has nonzero rational solutions. This equation has a one-dimensional space of rational solutions generated by $h = 1/(x^4 - x^3)$, so the corresponding solution of (11) is

$$u_1 = \frac{1}{x^2} + \frac{h'}{h} = -\frac{4x^2 - 4x + 1}{x^3 - x^2}.$$

Case $1/x^2 + 1/(x-1)$*:* Then we must do the change of variable $\overline{u} = 1/(1-x) + \tilde{u}$, and the Ricatti equation for $\tilde{u}$ is the one associated with

$$\overline{L} = B_2(\partial_x + \frac{1}{1-x})^2 + B_1(\partial_x + \frac{1}{1-x}) + B_0$$
$$= (x^5 - x^3)\partial_x^2 + (6x^4 - 5x^2 - x)\partial_x + 6x^3 - 3x - 3.$$

This equation has a one-dimensional space of rational solutions generated by $h = 1/x^3$, so the corresponding solution of (11) is

$$u_2 = \frac{1}{x^2} + \frac{1}{1-x} + \frac{h'}{h} = \frac{1}{x^2} + \frac{1}{1-x} - \frac{3}{x} = -\frac{4x^2 - 4x + 1}{x^3 - x^2}.$$

Since $u_1 = u_2$, it is the only rational solution. The corresponding exponential solution of (10) is

$$y = e^{\int u_1} = \frac{e^{-1/x}}{x^3(1-x)}$$

and there is only a 1-dimensional vector space of exponential solutions. The second solution is found by using this one to reduce the order and solving the resulting first order equation.

We now give a few examples solved with the AXIOM implementation of this algorithm. Since this implementation is also used as a top-level differential equation solver, the following modification is made: whenever m independent rational solutions $y_1, \ldots, y_m$ are found, we use reduction of order to reduce equation (1) to an equation of order $n - m$. This new equation can still have nonzero rational solutions (which are not rational solutions of the initial equation), so we can repeat the search for rational solutions until we get to an equation with no rational solutions. The solutions found in this way are always Liouvillian, but some may be non-exponential over $\overline{K}$ (because of the integration steps). The first two examples are from [5] and have complex singularitites which do not appear in the exponential solutions:

```
(x**4+x)*d(y x,x,2)+(x**3-1)*d(y x,x)-x**2*y x

        4       ,,          3       ,         2
(5)   (x  + x)y   (x) + (x  - 1)y (x) - x y(x)

-> solve(%,y,x).basis

   (6)
       +-------+
      3| 3
     [\|x  + 1 ,

       +-------+    x
      3| 3        ++              %F
      \|x  + 1  |    ------------------- d%F ]
                 ++              +-------+
                          3     3| 3
                       (%F  + 1)\|%F  + 1

-> 4*(x**2+x+1)**2*d(y x,x,2) -3*y x

         4     3      2             ,,
(11) (4x  + 8x  + 12x  + 8x + 4)y   (x) - 3y(x)

-> solve(%,y,x).basis

       +----------+              +----------+
       | 2              2x + 1   | 2
(12) [\|x  + x + 1 ,atan(------)\|x  + x + 1 ]
                           +-+
                          \|3
```

The highest-order example tried to date is from E.Tournier's thesis ([12]):

```
-> eq

   (2)
        6 (vi)          5 (v)           4 (iv)
      x y    (x) + 20x y   (x) + 122x y     (x)

   +
          3 ,,,          2 ,,
      258x y   (x) + 156x y  (x)

   +
          ,
      8xy (x) - 8y(x)

-> solve(%,y,x).basis

                             +---+
         1    log(x)    - \|- 1 log(x)
   (3)  [--,x,------,%e                  ,
          2      2
         x      x

            +---+            2
          \|- 1 log(x) log(x)  + log(x)
         %e                ,----------------]
                                 2
                                x
```

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